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AXIAL LOADING OF A RIGID DISC INCLUSION WITH A DEBONDED REGION

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Abstract—The present paper examines the problem of the axial loading of a penny-shaped inclusion which is embedded in partial bonded contact with an isotropic elastic solid of infinite extent. The debonded regions correspond to circular areas which are symmetrically and centrally located on the plane faces of the inclusion. The mathematical analysis of the problem focusses on the evaluation of the axial stiffness of the partially debonded inclusion. The mixed boundary value problem associated with the inclusion problem can be reduced to the solution of a single Fredholm integral equation of the second-kind. A numerical solution of this integral equation is used to generate the stiffness estimates for the axially loaded inclusion.

1. INTRODUCTION

The class of problems which deal with the stress analysis of elastic bodies reinforced with inclusions which are either rigid or elastic, is of importance to the study of muiltiphase composite materials. Detailed accounts of studies related to inclusion problems in classical elasticity are given by Eshelby (1961), Willis (1981), Walpole (1981) and Mura (1981). Flat disc shaped inclusions are a particular limiting case of the general class of three-dimensional ellipsoidal and spheroidal inclusions. The reinforcement of an elastic solid by disc-shaped inclusions enhances its stiffness and strength characteristics. A study by Wu (1966), indicated that disc-shaped inclusions give by far the most significant increase in the effective modulus of multiphase composites. Several investigators have therefore examined the disc inclusion problem related to an elastic medium of infinite extent in order to examine the influence of effects such as transverse isotropy of the medium, annular and elliptical configuration of inclusion, flexural behaviour of the inclusion, interaction with nuclei of strain, influences of traction-free surfaces, constrained surfaces and bi-material regions. The particular geometry of the disc inclusion enables the study of these problems by appeal to mixed boundary value problems related to a halfspace region. A comprehensive account of the disc inclusion problem in classical elasticity theory will be presented in a forthcoming article by Selvadurai (1989).

In the majority of studies relating to inclusion problems it is assumed that perfect continuity or a bonded contact exists at the inclusion–elastic medium interface. Researches of Ashby (1966), McClintock (1968), Argon *et al.* (1975) and others suggest that cavities can nucleate at the interfaces by tearing of the inclusion away from the ductile matrix or by cracking of a non-deformable inclusion. The category of problems which relate to partially bonded three-dimensional inclusions embedded in elastic media appear to have received only limited attention. Studies of flaws located at the boundary of cylindrical elastic inclusions embedded in elastic media with differing properties, are given by England (1966). Other classes of problems in which imperfect contacts are modelled by distributions of dislocations have been investigated by Bullough and Bilby (1956), Dundurs (1967) and Lin and Mura (1973). References to further studies are also given by Mura (1981). In the context of disc inclusion problems, Hunter and Gamblen (1974) and Keer (1975) have investigated problems related to disc inclusions in which complete debonding occurs at a plane face. In this particular paper we examine an axisymmetric problem related to a disc inclusion in which symmetric debonding exists over a circular region. Such delaminations can be induced by thickness non-uniformities of the disc inclusion. The debonded inclusion is loaded by a central force which acts in the axial direction. Also it is specifically assumed that the axial loading of the inclusion does not lead to the re-establishment of contact in the debonded regions. Alternatively, it may be assumed that the elastic medium is subjected to a homogeneous state of tensile stress normal to the plane of the inclusion. This state of stress can be assigned in such a way that the axial loading of the inclusion does not lead to the re-establishment of contact at the debonded region. The analysis focusses on the evaluation of the axial load-displacement relationship for the debonded inclusion. In the study of multiphase composite material behaviour, the reinforcing inclusions invariably interact with other defects such as cracks, dislocations, dipoles, centres of dilatation, etc. to alter the local energy field in the vicinity of the inclusion. This in return affects the properties of the solid. The solution developed in this paper for the directly loaded-debonded inclusion can be used in conjunction with Betti's reciprocal theorem to study the interaction of the inclusion with other nuclei of strain and external forces. A Hankel transform development of the governing equations is used to formulate the reduced mixed boundary value problem associated with the disc inclusion. The system of integral equations generated by the mixed boundary conditions is reduced to a single Fredholm integral equation of the second-kind. This integral equation is solved in a numerical fashion, to evaluate the load-displacement relationship for the debonded disc inclusion.

2. BASIC EQUATIONS

For the analysis of the axisymmetric problem related to the axial loading of the partially debonded rigid disc inclusion we employ strain potential approach proposed by Love (1927). In the absence of body forces, the solution of the displacement equations of equilibrium can be represented in terms of a bi-harmonic function $\Phi(r, z)$, i.e.

$$\nabla^2\nabla^2\Phi(r,z) = 0 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is the axisymmetric form of Laplace's operator referred to the cylindrical polar coordinate system. The components of the displacement vector $\mathbf{u}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ referred to the cylindrical polar coordinate system can be expressed in terms of the derivatives of Φ. We have

$$2Gu_r = -\frac{\partial^2\Phi}{\partial r\,\partial z} \tag{3}$$

$$2Gu_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} \tag{4}$$

where G and ν are the linear elastic shear modulus and Poisson's ratio, respectively. Similarly, the components of the stress tensor are given

$$\sigma_{rr} = \frac{\partial}{\partial z}\left\{\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\Phi \tag{5}$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left\{\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right\}\Phi \tag{6}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left\{(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\Phi \tag{7}$$

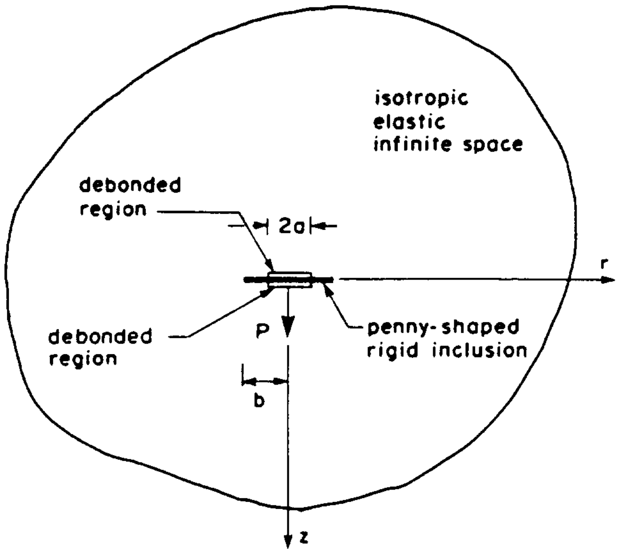

Fig. 1. Geometry of the debonded penny-shaped rigid inclusion embedded in an elastic infinite space.

$$\sigma_{rz} = \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\Phi \tag{8}$$

3. THE PARTIALLY DEBONDED INCLUSION PROBLEM

We examine the problem of a penny-shaped rigid inclusion of radius b which is embedded in bonded contact over the region $a \leqslant r \leqslant b$ where a is the radius of the symmetrically placed debonded regions (Fig. 1). The inclusion is displaced by an amount Δ in the z-direction. The force required to initiate this displacement is denoted by P. When the debonded regions remain so during the application of P, it can be shown (see Appendix A) that the particular mode of deformation induces a state of asymmetry about the plane $z = 0$. As a consequence, we can restrict the analysis to the examination of a single halfspace region occupying $z \geqslant 0$. The relevant mixed boundary conditions associated with the inclusion problem are as follows.

$$u_z(r,0) = \Delta; \quad a \leqslant r \leqslant b \tag{9}$$

$$u_r(r,0) = 0; \quad a \leqslant r < \infty \tag{10}$$

$$\sigma_{zz}(r,0) = 0; \quad 0 < r < a \tag{11}$$

$$\sigma_{zz}(r,0) = 0; \quad b < r < \infty \tag{12}$$

$$\sigma_{rz}(r,0) = 0; \quad 0 < r < a. \tag{13}$$

For the integral equation formulation of the mixed boundary value problem posed by (9)–(13) we seek solutions of (1) which can be obtained by a Hankel transform development of the basic differential equation (1). Furthermore the displacements and stress fields derived from $\Phi(r,z)$ should satisfy the regularity conditions $\mathbf{u} \to 0(1/R)$ and $\sigma \to 0(1/R^2)$ as $R(=[r^2+z^2]^{1/2}) \to \infty$.

Following Sneddon (1977), it can be shown that the relevant solution is given by

$$\Phi(r,z) = \int_0^\infty \xi[A(\xi)+zB(\xi)]\, e^{-\xi z} J_0(\xi r)\, d\xi \tag{14}$$

where $J_0(\xi r)$ is the zeroth-order Bessel function of the first hand. $A(\xi)$ and $B(\xi)$ are the arbitrary functions which are to be determined by satisfying the mixed boundary conditions (9)–(13). Employing the integral representation for $\Phi(r,z)$ given by (14) in the expressions for **u** and $\boldsymbol{\sigma}$ it can be shown that the mixed boundary conditions (9)–(13) can be reduced to the following system of integral equations.

$$H_0[\xi\{\xi A(\xi)+2(1-2\nu)B(\xi)\}, r] = -2G\Delta; \quad a \leqslant r \leqslant b \tag{15}$$

$$H_1[\xi\{-\xi A(\xi)+B(\xi)\}; r] = 0; \quad a \leqslant r < \infty \tag{16}$$

$$H_0[\xi^2\{\xi A(\xi)+(1-2\nu)B(\xi)\}; r] = 0; \quad 0 < r < a \tag{17}$$

$$H_0[\xi^2\{\xi A(\xi)+(1-2\nu)B(\xi)\}; r] = 0; \quad b < r < \infty \tag{18}$$

$$H_1[\xi^2\{\xi A(\xi)-2\nu B(\xi)\}; r] = 0; \quad 0 < r < a \tag{19}$$

where $H_n[g(\xi); r]$, $(n = 0, 1)$ is the Hankel transform of order n which is defined by

$$H_n[g(\xi); r] = \int_0^\infty \xi g(\xi) J_n(\xi r)\, d\xi \tag{20}$$

where $J_n(\xi r)$ is the nth order Bessel function of the first kind. To further reduce the system of integral equations (15)–(19) we make the assumption that as $a \to 0$ we should recover, from the solution developed, the appropriate result for the problem of the axial loading of the completely bonded rigid disc inclusion. We introduce functions $M(\xi)$ and $N(\xi)$ such that

$$A(\xi) = \frac{1}{2(1-\nu)\xi^3}\{-(1-2\nu)M(\xi)+N(\xi)\} \tag{21}$$

$$B(\xi) = \frac{1}{2(1-\nu)\xi^2}\{M(\xi)+N(\xi)\}. \tag{22}$$

Using these substitutions, the system of integral equations (15)–(19) can be reduced to the forms

$$H_0\left[\xi^{-1}\left\{N(\xi)+\frac{(1-2\nu)}{(3-4\nu)}M(\xi)\right\}; r\right] = -\frac{4G\Delta(1-\nu)}{(3-4\nu)}; \quad a \leqslant r \leqslant b \tag{23}$$

$$H_1[\xi^{-1}M(\xi); r] = 0; \quad a \leqslant r < \infty \tag{24}$$

$$H_0[N(\xi); r] = 0; \quad 0 < r < a \tag{25}$$

$$H_0[N(\xi); r] = 0; \quad b < r < \infty \tag{26}$$

$$H_1[\{(1-2\nu)N(\xi)-M(\xi)\}; r] = 0; \quad 0 < r < a. \tag{27}$$

Introduce an auxiliary function $\phi(t)$ such that

$$M(\xi) = \int_0^a \phi(t)\cos(\xi t)\,dt. \tag{28}$$

Substituting (28) into (24) we note that

$$\int_0^a \phi(t)\,dt = 0. \tag{29}$$

Integrating (27) it can be shown that

$$H_0[\xi^{-1}\{(1-2\nu)N(\xi)-M(\xi)\};r] = C_1; \quad 0<r<a \tag{30}$$

where C_1 is a constant. By using the substitution (28), the equation (30) can be reduced to an integral equation of the Abel type:

$$\int_0^r \frac{\phi(t)\,dt}{[r^2-t^2]^{1/2}} = -C_1+(1-2\nu)\int_0^\infty N(\xi)J_0(\xi r)\,d\xi; \quad 0<r<a. \tag{31}$$

The solution of (31) can be written as

$$\phi(t) = -\frac{2}{\pi}C_1+\frac{2(1-2\nu)}{\pi}\int_0^\infty N(\xi)\cos(\xi t)\,d\xi; \quad 0<t<a. \tag{32}$$

The value of the constant C_1 can be determined by making use of (32) and the consistency condition (29); consequently, the complete expression for (32) takes the form

$$\phi(t) = \frac{2}{\pi}(1-2\nu)\left[\int_0^\infty N(\xi)\left\{\cos(\xi t)-\frac{\sin(a\xi)}{a\xi}\right\}d\xi\right]; \quad 0<t<a. \tag{33}$$

We now examine the system of triple integral equations defined by (23), (25) and (26). Introduce an auxiliary function $g(r)$ such that:

$$\int_0^\infty \xi N(\xi)J_0(\xi r)\,d\xi = g(r); \quad a<r<b. \tag{34}$$

Using the properties of Hankel transforms we can obtain an integral expression for $N(\xi)$; as a result, (23) can be expressed in the form

$$\int_a^b ug(u)L(u,r)\,du+\frac{(1-2\nu)}{(3-4\nu)}\int_0^\infty M(\xi)J_0(\xi r)\,d\xi = -\frac{4G\Delta(1-\nu)}{(3-4\nu)}; \quad a\leqslant r\leqslant b \tag{35}$$

where the operator $L(u,r)$ is defined by

$$L(u,r) = \int_0^\infty J_0(\xi u)J_0(\xi r)\,d\xi = \frac{2}{\pi}\int_0^{\min(u,r)}\frac{ds}{[(u^2-s^2)(r^2-s^2)]^{1/2}} \tag{36}$$

and min (u,r) denotes the minimum values of u and r. The first integral in (35) can be written as:

$$\int_a^b \mathrm{d}u \int_0^{\min(u,r)} \mathrm{d}s = \int_a^r \mathrm{d}u \int_0^u \mathrm{d}s + \int_r^b \mathrm{d}u \int_0^r \mathrm{d}s$$

$$= \int_a^r \mathrm{d}s \int_s^b \mathrm{d}u + \int_0^a \mathrm{d}s \int_a^b \mathrm{d}u. \tag{37}$$

Using the properties of these integrals and observing that

$$\int_0^\infty M(\xi) J_0(\xi r)\, \mathrm{d}\xi = \int_0^a \frac{\phi(u)\, \mathrm{d}u}{[r^2 - u^2]^{1/2}} \tag{38}$$

we obtain, from (35), the following

$$\int_a^r \frac{\mathrm{d}s}{[r^2-s^2]^{1/2}} \int_s^b \frac{ug(u)\, \mathrm{d}u}{[u^2-s^2]^{1/2}} = -\int_0^a \frac{\mathrm{d}s}{[r^2-s^2]^{1/2}} \int_a^b \frac{ug(u)\, \mathrm{d}u}{[u^2-s^2]^{1/2}}$$

$$-\frac{2\pi G\Delta(1-\nu)}{(3-4\nu)} - \frac{\pi}{2} \frac{(1-2\nu)}{(3-4\nu)} \int_0^a \frac{\phi(u)\, \mathrm{d}u}{[r^2-u^2]^{1/2}}; \quad a \leqslant r \leqslant b \tag{39}$$

Introduce the substitution

$$\int_s^b \frac{ug(u)\, \mathrm{d}u}{[u^2-s^2]^{1/2}} = T(s); \quad a \leqslant s \leqslant b. \tag{40}$$

Note that (40) is an integral equation of the Abel type, the solution of which takes the form

$$g(u) = -\frac{2}{\pi u} \frac{\mathrm{d}}{\mathrm{d}u} \int_u^b \frac{sT(s)\, \mathrm{d}s}{[s^2-u^2]^{1/2}}; \quad a \leqslant u \leqslant b. \tag{41}$$

Introducing (40) into (39) we obtain a second Abel-type integral equation for $T(s)$; its solution gives the following (see also the results given by Cooke, 1963):

$$T(s) = -\frac{4G\Delta(1-\nu)s}{(3-4\nu)[s^2-a^2]^{1/2}} - \frac{4s}{\pi^2[s^2-a^2]^{1/2}} \int_a^b \frac{tT(t)K(s,t)\, \mathrm{d}t}{[t^2-a^2]^{1/2}}$$

$$-\frac{(1-2\nu)s}{(3-4\nu)[s^2-a^2]^{1/2}} \int_0^a \frac{[a^2-u^2]^{1/2}\phi(u)\, \mathrm{d}u}{(s^2-u^2)}; \quad a \leqslant s \leqslant b \tag{42}$$

where

$$K(s,t) = \int_0^a \frac{(a^2-y^2)\, \mathrm{d}y}{[(s^2-y^2)(t^2-y^2)]}. \tag{43}$$

Making use of (33), the expression for $N(\xi)$ derived from (34), and (41), we obtain the following integral equation for $\phi(u)$.

$$\frac{-(1-2\nu)s}{(3-4\nu)[s^2-a^2]^{1/2}} \int_0^a \frac{[a^2-u^2]^{1/2}\phi(u)\, \mathrm{d}u}{[s^2-u^2]} = \frac{4s}{\pi^2[s^2-a^2]^{1/2}} \int_a^b tL^*(s,t)T(t)\, \mathrm{d}t \tag{44}$$

where

$$L^*(s,t) = \frac{(1-2\nu)^2}{(3-4\nu)}\left[\int_0^a \left\{\left(\frac{\pi}{2a} - \frac{1}{a^2-u^2}\right)\frac{1}{(t^2-a^2)^{1/2}}\right.\right.$$
$$\left.\left. - \int_a^t \left(\frac{1}{p[p^2-a^2]^{1/2}} - \frac{2p}{[p^2-u^2]^2}\right)\frac{dp}{[t^2-p^2]^{1/2}}\right\}\frac{[a^2-u^2]^{1/2}}{[s^2-u^2]}\,du\right]. \quad (45)$$

Introducing the transformations

$$T(s) = -\frac{4G\Delta(1-\nu)}{(3-4\nu)}T^*(s), \quad (46)$$

the integral equations (44) and (45) can be reduced to a single integral equation for $T^*(s)$:

$$T^*(s) = \frac{s}{[s^2-a^2]^{1/2}} + \frac{4s}{\pi^2[s^2-a^2]^{1/2}}\int_a^b tT^*(t)\left[L^*(s,t) - \frac{K(s,t)}{[t^2-a^2]^{1/2}}\right]dt; \quad a \leqslant s \leqslant b. \quad (47)$$

Also, using the substitutions

$$s = a\sec\Theta; \quad t = a\sec\omega; \quad (\sec^2\Theta)T^*(a\sec\Theta) = H(\Theta), \quad (48)$$

eqn (47) can be written in the form

$$H(\Theta)\sin\Theta\cos^2\Theta = 1 + \frac{4}{\pi^2}\int_0^{\sec^{-1}(b/a)} H(\omega)[\tilde{L}(\Theta,\omega) - \tilde{K}(\Theta,\omega)]\,d\omega; \quad 0 \leqslant \Theta \leqslant \sec^{-1}(b/a) \quad (49)$$

where

$$\tilde{K}(\Theta,\omega) = \frac{1}{(\sec^2\Theta - \sec^2\omega)}[\sin^2\omega\sec\omega\ln\{\tan(\omega/2)\} - \sin^2\Theta\sec\Theta\ln\{\tan(\Theta/2)\}] \quad (50)$$

$$\tilde{L}(\Theta,\omega) = \frac{(1-2\nu)^2a^2\tan\omega}{(3-4\nu)}\left[\frac{1}{a\tan\omega}\left\{\frac{\pi^2}{4a}(1-\sin\Theta) - \frac{\pi}{2a^2\tan\Theta\sec\Theta}\right\}\right.$$
$$\left. + \int_0^a \frac{(a^2-u^2)^{1/2}\,du}{(u^2-a^2\sec^2\Theta)}\int_a^{a\sec\omega}\left\{\frac{1}{p[p^2-a^2]^{1/2}} - \frac{2p}{[p^2-u^2]^2}\right\}\frac{dp}{(a^2\sec^2\omega - p^2)^{1/2}}\right]. \quad (51)$$

It may be noted that when $\Theta \to \omega$, $\tilde{K}(\Theta,\omega)$ can be evaluated by applying L'Hospital's rule. The integral equation (49) is a Fredholm integral equation of the second-kind for the function $H(\Theta)$. The integral equation (49) can be solved in a numerical fashion to derive results of engineering interest. In the present paper we shall focus attention on the evaluation of the axial load–displacement relationship for the partially bonded disc inclusion.

Considering (25), we note that

$$\sigma_{zz}(r,0) = \int_0^\infty \xi N(\xi)J_0(\xi r)\,d\xi = \frac{1}{r}\frac{d}{dr}\left\{r\int_0^\infty N(\xi)J_1(\xi r)\,d\xi\right\}; \quad a < r < b. \quad (52)$$

Using (34) and (41), the above result can be reduced to the form

$$\sigma_{zz}(r,0) = -\frac{2}{\pi r}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^b \frac{sT(s)\,\mathrm{d}s}{[s^2-r^2]^{1/2}}; \quad a < r < b. \tag{53}$$

Considering the axial contact stresses at the bonded surfaces of the inclusion it is evident that $\sigma_{zz}(r,0) = -\sigma_{zz}(r,0^-)$ where the negative superscript refers to the contact stresses at the surface of the inclusion–elastic medium interface in contact with the halfspace region $z \leqslant 0$. Considering the equilibrium of the inclusion we obtain:

$$P = -\int_0^{2\pi}\int_a^b [\sigma_{zz}(r,0^+)-\sigma_{zz}(r,0^-)]r\,\mathrm{d}r\,\mathrm{d}\Theta. \tag{54}$$

Using the results of (46), (48) and (53), (54) can be reduced to the form

$$P = \frac{32G\Delta(1-\nu)a}{(3-4\nu)}\int_0^{\sec^{-1}(b/a)} H(\Theta)\,\mathrm{d}\Theta. \tag{55}$$

4. NUMERICAL SOLUTION OF THE INTEGRAL EQUATION

In the ensuing we shall present a brief summary of the numerical procedures that are used to solve the Fredholm integral equation of the second-kind derived previously. More complete accounts of the various procedures that can be employed in the solution of this category of integral equation are summarized by Atkinson (1976) and Baker (1977). The Fredholm integral equation (49) can be written in the form

$$H(\Theta)\sin\Theta\cos^2\Theta - \frac{4}{\pi^2}\int_0^{\zeta} H(\omega)[\tilde{L}(\Theta,\omega)-\tilde{K}(\Theta,\omega)]\,\mathrm{d}\omega = 1; \quad 0 \leqslant \Theta \leqslant \zeta \tag{56}$$

where $\zeta = \cos^{-1}(a/b)$; the function $\tilde{K}(\Theta,\omega)$ is defined by (50), the function $\tilde{L}(\Theta,\omega)$ can be written in the modified form

$$\tilde{L}(\Theta,\omega) = \frac{(1-2\nu)^2}{(3-4\nu)}\left[\frac{\pi^2}{4}(1-\sin\Theta)(1-\sin\omega) - \frac{\pi}{2a}\frac{\cos^2\Theta}{\sin\Theta}\right.$$
$$\left. + \frac{\pi}{2a}\frac{\tan^2\omega}{(\sec^2\omega-\sec^2\Theta)}\left\{\frac{\cos^2\Theta}{\sin\Theta} - \frac{\cos^2\omega}{\sin\omega}\right\} + \frac{\tan\omega}{a}I_0(\sec\Theta,\sec\omega)\right] \tag{57}$$

and the integral function $I_0(\alpha,\beta)$ is given by

$$I_0(\alpha,\beta) = \int_0^1 \frac{[1-\zeta^2]^{1/2}}{(\zeta^2-\alpha^2)(\beta^2-\zeta^2)^{1/2}}\ln\left|\frac{[1-\zeta^2]^{1/2}}{[\beta^2-\zeta^2]^{1/2}+[\beta^2-1]^{1/2}}\right|\mathrm{d}\zeta. \tag{58}$$

In the numerical procedure adopted here, we employ a Gaussian quadrature scheme to solve the integral equation (56). Considering N Gaussian points, (56) can be reduced to a matrix equation of the form

$$\mathbf{K}_{ij}\mathbf{H}(\Theta_j) = \mathbf{1} \tag{59}$$

with $i,j = 1,2,\ldots,N$.

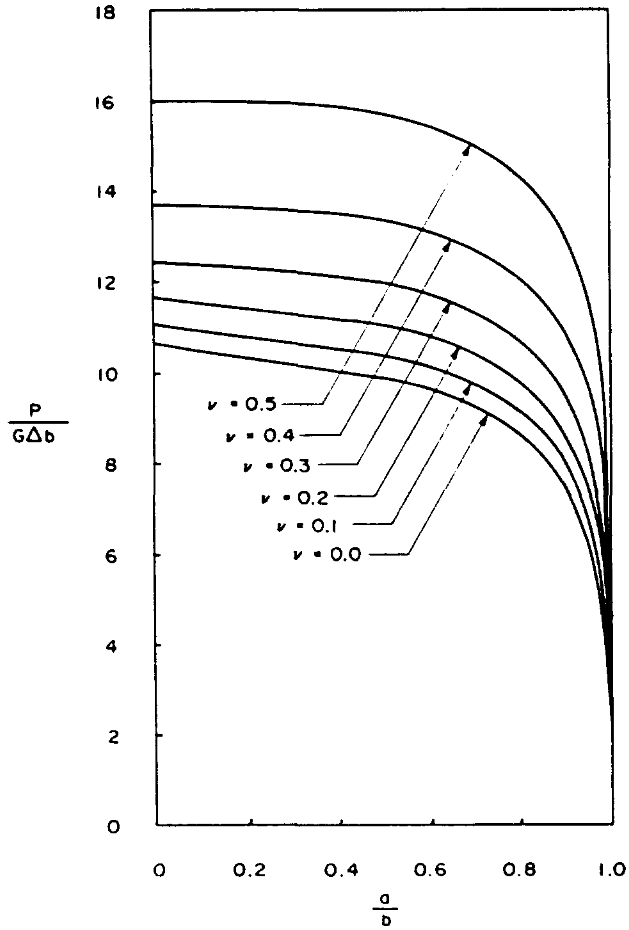

Fig. 2. The stiffness of debonded penny-shaped inclusion.

$$\Theta_i = \frac{\zeta}{2}(1+g_i)$$

$$K_{ij} = \begin{cases} -\dfrac{4}{\pi^2}[\tilde{L}(\Theta_i, \Theta_j) - \tilde{K}(\Theta_i, \Theta_j)]\lambda_j \dfrac{\zeta}{2}; & i \neq j \\ \sin\Theta_i \cos^2\Theta_i - \dfrac{4}{\pi^2} f(\Theta_i)\lambda_i \dfrac{\zeta}{2}; & i = j \end{cases} \tag{60}$$

where g_i and λ_i are, respectively, the points and weights of the quadrature scheme and $f(\Theta)$ is obtained from the result $\underset{\Theta \to \omega}{Lt}\,[\tilde{L}(\Theta, \omega) - \tilde{K}(\Theta, \omega)]$; i.e.

$$f(\Theta) = \frac{(1-2\nu)^2}{(3-4\nu)}\left\{\frac{\pi^2}{4}(1-\sin\Theta)^2 - \frac{\pi\cos^4\Theta}{4a\sin\Theta} + \frac{\tan\Theta}{a} I_0(\sec\Theta, \sec\omega)\right\}. \tag{61}$$

Upon solution of the matrix equation (59), the relevant load-displacement relationship (55) can be evaluated in the discretized form

$$\frac{P}{G\Delta b} = \frac{32(1-\nu)}{(3-4\nu)}\left\{\frac{\zeta a}{2b}\sum_{i=1}^{N} H(\Theta_i)\lambda_i\right\}. \tag{62}$$

5. NUMERICAL RESULTS AND CONCLUSIONS

The numerical technique outlined in the previous section is used to evaluate the axial load-displacement relationship for the partially debonded penny-shaped rigid inclusion embedded in an elastic infinite space. Figure 2 illustrates the manner in which the axial stiffness of the embedded inclusion is influenced by the extent of the debonded region and Poisson's ratio of the elastic medium. A total of 24 Gauss points were used in the numerical evaluation. An increase of the number of points from 24 to 64 did not result in any appreciable improvements of these numerical results. The numerical results also converge to the exact closed form result (see e.g. Selvadurai, 1976) as $(a/b) \rightarrow 0$; i.e. $P/G\Delta b = 32(1-\nu)/(3-4\nu)$. The numerical results also indicate that appreciable changes in the axial stiffness of the debonded penny-shaped inclusion occur only for values of $(a/b) > 0.4$.

The paper outlines the mathematical analysis of the axial loading of a penny-shaped rigid inclusion which is embedded in partial bonded contact with an isotropic elastic infinite space. Since the debonding is assumed to be symmetric, the infinite space problem can be effective reduced to a mixed boundary value problem associated with a halfspace region. The analysis can also be extended to situations in which debonding occurs in a non-symmetric fashion. Such an analysis however requires the consideration of two sets of mixed boundary value problems for the halfspace regions $z > 0$ and $z < 0$. For this case the symmetry constraints on u_r and σ_{zz} in the region $z > 0$; $r \in (b, \infty)$ are replaced by continuity conditions for u_r, u_z, σ_{zz} and σ_{rz}.

It is important to note that the state of stress at the boundary of the rigid inclusion and at the boundary of the debonded region are singular. In particular it is known (Atkinson, 1979) that the stress singularity at the boundary of such debonded regions is oscillatory. Consequently, in situations where the exact stress distributions or the stress intensity factors at the inner debonded boundary are required it is necessary to perform the analysis by appeal to a formulation based on the Hilbert problem where the exact nature of the oscillatory singularity is invoked. On the other hand, if the results of primary interest focus on the evaluation of global estimates such as load displacement responses for the inclusion then the integral transform based scheme, which is adopted in the present paper, yields accurate results. The justification for this simplification can be provided by examining the axisymmetric problem of the adhesively bonded punch on a halfspace region. It can be shown that the stiffness of the punch derived via the integral transform-based approach can be evaluated to within 0.6% of the exact solution for the extreme cases when $\nu = 0$. When the material is incompressible both approaches yield the same result. The methodology presented in this paper, therefore, provides a useful procedure for the determination of the stiffness characteristics of partially bonded rigid inclusions which are embedded in elastic media.

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APPENDIX A

Consider the full space with region (1) corresponding to $z \geqslant 0$ and region (2) corresponding to $z \leqslant 0$. The appropriate integral expressions of Love's strain potential for regions (1) and (2) take the forms

$$\varphi_1 = \int_0^\infty \xi[A(\xi)+B(\xi)z]\,e^{-\xi z}\,J_0(\xi r)\,d\xi \tag{A1}$$

$$\varphi_2 = \int_0^\infty \xi[C(\xi)+D(\xi)z]\,e^{\xi z}\,J_0(\xi r)\,d\xi. \tag{A2}$$

Considering the full space region, the mixed boundary conditions at $z = 0$, posed by the disbonded inclusion problem take the following forms.

$$u_z^{(1)} = \Delta;\quad a \leqslant r \leqslant b \tag{A3}$$

$$u_z^{(2)} = \Delta;\quad a \leqslant r \leqslant b \tag{A4}$$

$$u_r^{(1)} = 0;\quad a \leqslant r \leqslant b \tag{A5}$$

$$u_r^{(2)} = 0;\quad a \leqslant r \leqslant b \tag{A6}$$

$$\sigma_{zz}^{(1)} = \sigma_{zz}^{(2)} = 0;\quad 0 < r < a \tag{A7}$$

$$\sigma_{rz}^{(1)} = \sigma_{rz}^{(2)} = 0;\quad 0 < r < a \tag{A8}$$

$$u_r^{(1)} = u_r^{(2)};\quad b \leqslant r < \infty \tag{A9}$$

$$u_z^{(1)} = u_z^{(2)};\quad b \leqslant r < \infty \tag{A10}$$

$$\sigma_{zz}^{(1)} = \sigma_{zz}^{(2)};\quad b < r < \infty \tag{A11}$$

$$\sigma_{rz}^{(1)} = \sigma_{rz}^{(2)};\quad b < r < \infty. \tag{A12}$$

For the region (1) ($z > 0$) we have

$$2Gu_z^{(1)}(r,0) = -\int_0^\infty \xi[\xi A(\xi)+2(1-2\nu)B(\xi)]J_0(\xi r)\,d\xi \tag{A13}$$

$$2Gu_r^{(1)}(r,0) = \int_0^\infty \xi[-\xi A(\xi)+B(\xi)]J_1(\xi r)\,d\xi \tag{A14}$$

$$\sigma_{zz}^{(1)}(r,0) = \int_0^\infty \xi^2[\xi A(\xi+(1-2\nu)B(\xi)]J_0(\xi r)\,d\xi \tag{A15}$$

$$\sigma_{rz}^{(1)}(r,0) = \int_0^\infty \xi^2[\xi A(\xi)-2\nu B(\xi)]J_1(\xi r)\,d\xi. \tag{A16}$$

Similarly for the region (2) ($z < 0$) we have

$$2Gu_z^{(2)}(r,0) = \int_0^\infty \xi[\xi C(\xi) - 2(1-2\nu)D(\xi)]J_0(\xi r)\,\mathrm{d}\xi \tag{A17}$$

$$2Gu_r^{(2)}(r,0) = \int_0^\infty \xi[D(\xi) + \xi C(\xi)]J_1(\xi r)\,\mathrm{d}\xi \tag{A18}$$

$$\sigma_{zz}^{(2)}(r,0) = \int_0^\infty \xi^2[(1-2\nu)D(\xi) - \xi C(\xi)]J_0(\xi r)\,\mathrm{d}\xi \tag{A19}$$

$$\sigma_{rz}^{(2)}(r,0) = \int_0^\infty \xi^2[\xi C(\xi) + 2\nu D(\xi)]J_1(\xi r)\,\mathrm{d}\xi. \tag{A20}$$

For the present let us assume that $u_z^{(1)} = u_z^{(2)}$ for $r \in (0, \infty)$. Then from (A13) and (A17) we have

$$A(\xi) = -C(\xi); \quad B(\xi) = D(\xi). \tag{A21}$$

Using (A21) and (A13)–(A20) we have

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0) \tag{A22}$$

$$u_r^{(1)}(r,0) = -u_r^{(2)}(r,0) \tag{A23}$$

$$\sigma_{zz}^{(1)}(r,0) = -\sigma_{zz}^{(2)}(r,0) \tag{A24}$$

$$\sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0). \tag{A25}$$

Using (A22)–(A25), the boundary conditions (A3)–(A12) can be reduced to the forms

$$u_z^{(1)}(r,0) = \Delta; \quad a < r < b \tag{A26}$$

$$u_r^{(1)}(r,0) = 0; \quad a \le r < \infty \tag{A27}$$

$$\sigma_{rz}^{(1)}(r,0) = 0; \quad 0 < r < a \tag{A28}$$

$$\sigma_{zz}^{(1)}(r,0) = 0; \quad 0 < r < a \tag{A29}$$

$$\sigma_{zz}^{(1)}(r,0) = 0; \quad b < r < \infty. \tag{A30}$$

These are exactly the same as the boundary conditions given in eqns (9)–(12) of the paper. In obtaining (A26)–(A30) we have assumed that $u_z^{(1)}(r,0) = u_z^{(2)}(r,0)$ for $r \in (0, \infty)$. From (A3), (A4) and (A10) we note that $u_z^{(1)}(r,0) = u_z^{(2)}(r,0)$ for both $r \in (a, b)$ and $r \in (b, \infty)$. Consequently, the assertion is accurate provided

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0) \quad 0 < r < a.$$

Consider the displacement Δ of the inclusion as shown in Fig. A1(a). Let the surfaces of the disbonded region $r \in (0, a)$ exhibit displacements $\bar{u}_z^{(1)}$ and $\bar{u}_z^{(2)}$ at the respective regions. When the inclusion is displaced as shown in Fig. A1(b) the associated surface displacements at the debonded regions are $-\bar{u}_z^{(1)}$ and $-\bar{u}_z^{(2)}$. Figure A1(c) is obtained by a rigid body rotation of Fig. A1(b), about the x-axis and it may be noted that since the halfspace regions (1) and (2) are identical the designations could be interchanged. Consequently $u_z^{(1)} = u_z^{(2)}$ for $r \in (0, a)$. Thus the reduced boundary value problem pertaining to a halfspace region as defined by the mixed boundary conditions (9)–(12) in the paper is the complete representation of the full space problem.

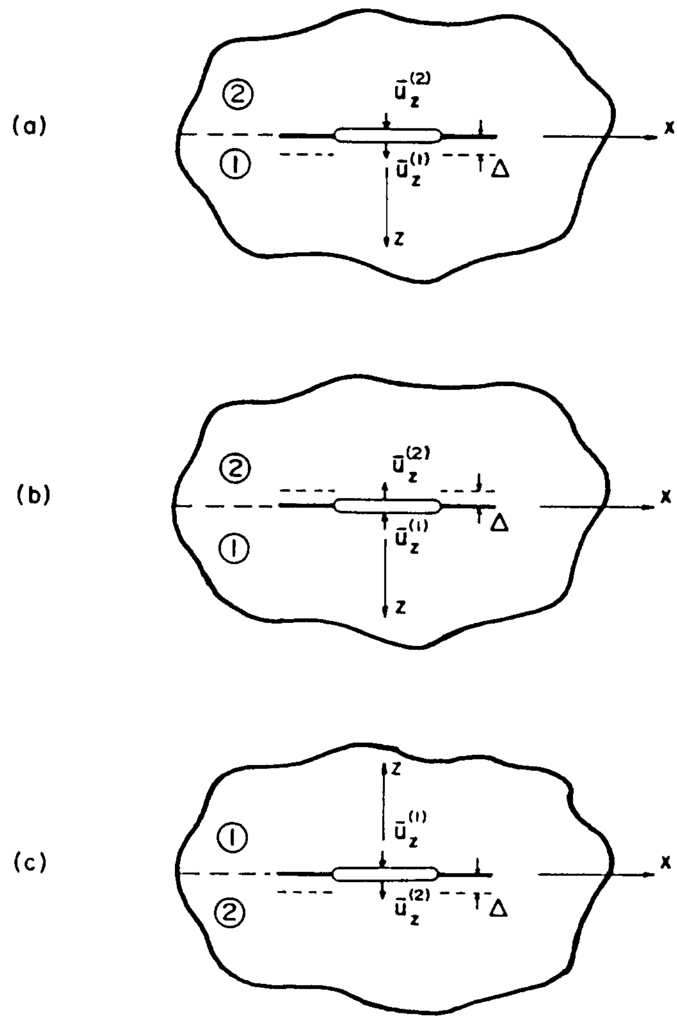

Fig. A1. Reduction of the boundary conditions applicable to the debonded inclusion problem.